

Chaotic inflation from a scalar field in non-classical states

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Abstract

We study chaotic inflation driven by a real, massive, homogeneous minimally coupled scalar field in a flat Robertson-Walker spacetime. The semiclassical limit for gravity is considered, whereas the scalar field is treated quantum mechanically by the technique of invariants in order to also investigate the dynamics of the system for non-classical states of the latter. An inflationary stage is found to be possible for a large set of initial quantum states, obviously including the coherent ones. States associated with a vanishing mean value of the field (such as the vacuum and the thermal) can also lead to inflation, however for such states we cannot make a definitive prediction due to the importance of higher order corrections during inflation. The results for the above coupled system are described and their corrections evaluated perturbatively.

1 Introduction

A cosmological exponential expansion was proposed in the late 70's and early 80's in order to solve the problems of the big bang model [1, 2]. Many proposals have been made for the realization of such an expansion, called inflation. The chaotic model proposed by Linde [3] was devoted to searching for conditions of a scalar field for which inflation may occur. No restrictions, such as an initial state of thermal equilibrium or a minimum of an effective potential as in previous models [2, 4], were assumed. Indeed the strength of chaotic models is their relative insensitivity to initial conditions[5]. In this sense they are more generic than other models, such as new inflation [4], which are more dependent on initial conditions and/or require fine tuning of the parameters which enter in the potential of the scalar field. However, when chaotic models were first presented, it seemed that an implausible initial homogeneity was required in order for inflation to start. Therefore much effort was dedicated to showing, by numerical and/or analytical estimates, that anisotropies, small and large inhomogeneities will in general not suppress the onset of inflation (see the review [6] for a detailed discussion of this topic).

In this work, limiting ourselves to the homogeneous case, we examine the idea of chaotic inflation from a semiclassical point of view, in order to enlarge the conditions which allow for inflation to arise from quantum mechanical initial states. Indeed, most of the chaotic models are also based on a classical behaviour for the scalar field. The "classicality" of the scalar field is a rather questionable assumption: let us note that the realization of an inflationary domain, usually as large as several horizon radii [6], implies that one is considering a region of space-time characterized by a typical size much smaller than the Compton wavelength ($1/m$) of the scalar field ($m \ll H$ is usually considered in order to obtain sufficiently small density perturbations during inflation [7, 8]). On considering gravitational interactions it is difficult to think of the scalar field as being in a classical state when gravity effects it on distances smaller than its Compton wavelength.

For the above reasons, it is of interest to study the evolution of the system for non classical states of the scalar field also. Here we present a quantum mechanical treatment of

the homogeneous scalar field which drives a chaotic inflation model with a quadratic potential associated with its mass, $V(\phi) = m^2 \phi^2/2$. This is greatly facilitated by the possibility of obtaining a Fock space of exact solutions to the Schrödinger equation for the matter field by solving the related harmonic oscillator problem with time dependent parameters, using the technique of invariants [10]. Our intention is to investigate whether quantum states of the scalar field exist which lead to a sufficient inflationary phase.

In a certain sense we are analyzing "phenomenologically" the inflationary stage which is originated by a variety of possible quantum states. Therefore our conclusions could be of interest also for quantum cosmology, whose goal is the initial distribution of the scalar field after the quantum era [11]. A classical initial distribution was considered in most of the cases treated in literature, except for some rare examples where quantum uncertainty was used to suggest a spread-out distribution for the classical value [12].

We shall find an inflationary stage, sufficient to solve the problems of standard cosmology, also for states different from coherent ones [13, 14]. In fact there is also the possibility that states such as the vacuum or a thermal ensemble (which have a zero expectation value for the scalar field) lead to an inflationary stage for which the time derivative of the Hubble parameter H is constant and negative. Indeed the time behaviour for H , which satisfies an interesting effective equation, is the same as previously found for the classical coupled system during inflation [15]. However we are not able to study the evolution of the above non-classical states for arbitrary times because of quantum corrections (due to fluctuations) to the usual equations for matter.

The above mentioned fluctuations arise naturally from the Born-Oppenheimer (BO) reduction [16, 17], which we use to study the evolution of the matter-gravity system. In some cases these quantum corrections are not small and force the scalar field out of the initial quantum state. It is not clear whether this effect also alters the above mentioned qualitative behaviour of the scale factor during inflation.

The outline of the paper is as follows: we set up our formalism in section 2 which is divided into four paragraphs, of which the first two treat the Wheeler-De Witt (WDW) equation [18] for the Robertson-Walker (RW) minisuperspace considered and its BO reduction. In the third paragraph we describe the method of invariants for time dependent harmonic oscillators and in the last we estimate the corrections to the usual matter and gravity equations of motions while relegating some useful formulae for their computation to the appendix. It is clear that in doing this we are taking seriously the WDW equation, which represents the full quantum version of this homogeneous cosmological toy model. The inclusion of all the Fourier modes of the scalar field would certainly represent a more reliable version of what is being studied here. However, we think that consideration of an exact quantum problem (although a finite dimensional one rather than an infinite), from which the semiclassical approximation can be obtained and verified is an important point worth studying. Section 3 is devoted to the study of the dynamical system. In the fourth section we present the numerical analysis and summarize our results in the Conclusions.

2 Basic equations.

2.1 The classical system

We shall study one of the simplest cases of chaotic inflation, which is based on the theory of a real free massive scalar field ϕ minimally coupled to gravity. The action is:

$$S \equiv \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \quad (1)$$

where \mathcal{L} is the lagrangian density and m is the inverse Compton wavelength of the field ϕ . Further we consider a RW line element with flat spatial section:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(t) dt^2 + a^2(t) d\vec{x}^2 \quad (2)$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor. On requiring homogeneity for the scalar field, we obtain through an Arnowitt-Deser-Misner decomposition, the following

form of the action (1):

$$S = \int dt d\vec{x} N a^3 \left[\frac{1}{16\pi G} \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right) + \frac{1}{2} \left(\frac{1}{N^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - m^2 \phi^2 \right) \right]. \quad (3)$$

where we have neglected boundary terms. The *three-curvature* scalar ${}^{(3)}R$, the square of the *extrinsic curvature tensor* K_{ij} and the *extrinsic curvature scalar* K for the metric line element (2) are respectively:

$${}^{(3)}R = 0, \quad K_{ij} K^{ij} = \frac{3}{N^2} \frac{\dot{a}^2}{a^2}, \quad K = -\frac{3}{N} \frac{\dot{a}}{a}, \quad (4)$$

where a dot denotes the derivative with respect to the time t . By using the relations (4) in the expression for the action (3) we obtain the following lagrangian:

$$L = N V \left[-\frac{M_{pl}^2}{2} a \frac{\dot{a}^2}{N^2} + \frac{a^3}{2} \left(\frac{\dot{\phi}^2}{N^2} - m^2 \phi^2 \right) \right], \quad (5)$$

where V is the volume of three space, $M_{pl}^2 = 3/4\pi G$ and N plays the role of a Lagrange multiplier and is not a dynamical variable. The Hamiltonian is given by

$$H_{tot} = N V \left[-\frac{1}{2M_{pl}^2} \frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{2a^3} + a^3 \frac{m^2}{2} \phi^2 \right] \equiv N V \left[-\frac{1}{2M_{pl}^2} \frac{\pi_a^2}{a} + H_M \right], \quad (6)$$

where $\pi_a = -M_{pl}^2 a \dot{a}/N$ and $\pi_\phi = a^3 \dot{\phi}/N$ are the momenta conjugate to a and ϕ respectively. The presence of N as a factor in (6) reveals that the total Hamiltonian has to be zero: this equation is usually called *Hamiltonian constraint*

$$\frac{\partial H_{tot}}{\partial N} = 0 \quad (7)$$

and describes the dynamics, while the so called *momentum constraints* are trivially zero in this minisuperspace model. In order to quantize the system we rescale $a \rightarrow aV^{1/3}$ absorbing the V factor into the Hamiltonian. Henceforth we shall work with an a having the canonical dimension of length.

2.2 The quantum system and its BO reduction

We start from a fully quantized gravity-matter system and subsequently return to a system in which gravity is semi-classical [19] and coupled to quantized matter, as is illustrated in [1, 17]. With a coherent state [13, 14] for matter it is also possible to return to the completely classical case.

On quantizing the Hamiltonian constraint one obtains the WDW equation [18]:

$$\hat{H}_{tot} \Psi(a, \phi) \equiv \left(\frac{\hbar^2}{2M_{pl}^2} \frac{\partial^2}{\partial a^2} \frac{1}{a} + \hat{H}_M \right) \Psi(a, \phi) = 0 \quad (8)$$

where Ψ is a function of a and ϕ and describes both gravity and matter. Subsequently one performs a BO decomposition of Ψ as:

$$\Psi(a, \phi) = a \psi(a) \chi(a, \phi) \quad (9)$$

where $\chi(a, \phi)$ is not further separable. Let us add a brief comment on the quantization of the Hamiltonian constraint and on the assumption (9): in order to obtain eq. (8) from eq. (7) a particular choice of factor ordering has been made ($\frac{\pi_a^2}{a} \rightarrow \hat{\pi}_a^2 \frac{1}{a}$) [9]. The ordering ambiguity in this context has been previously noted [11]. We observe that our choice is convenient since it is associated with eq. (9), which agrees with $\Psi = 0$ for $a \leq 0$ as expected. All orderings will coincide in the classical limit and differ by $\mathcal{O}(\hbar)$. We have verified that such differences

are negligible during the inflationary regime and in the semiclassical limit for gravity, which we consider.

An effective equation of motion for ψ may now be obtained [17] by first substituting the above decomposition into eq. (8), contracting with χ^* and dividing by $\langle \chi | \chi \rangle$. One then has :

$$\begin{aligned} \left[\frac{\hbar^2}{2M_{pl}^2} D^2 + a \frac{\langle \chi | \hat{H}_M | \chi \rangle}{\langle \chi | \chi \rangle} \right] \psi &= - \frac{\hbar^2}{2M_{pl}^2} \frac{\langle \chi | \bar{D}^2 | \chi \rangle}{\langle \chi | \chi \rangle} \psi \\ &= \frac{\hbar^2}{2M_{pl}^2} \frac{1}{\langle \chi | \chi \rangle} \left\{ \langle \chi | \bar{\partial} \left(1 - \frac{|\chi\rangle\langle\chi|}{\langle \chi | \chi \rangle} \right) \frac{\partial}{\partial a} | \chi \rangle \right\} \psi \end{aligned} \quad (10)$$

where we have introduced covariant derivatives:

$$D \equiv \frac{\partial}{\partial a} + i\mathcal{A}; \quad \bar{D} \equiv \frac{\partial}{\partial a} - i\mathcal{A} \quad (11)$$

with:

$$\mathcal{A} \equiv -i \frac{\langle \chi | \frac{\partial}{\partial a} \chi \rangle}{\langle \chi | \chi \rangle} \equiv -i \langle \frac{\partial}{\partial a} \rangle \quad (12)$$

and a scalar product:

$$\langle \chi | \chi \rangle \equiv \int d\phi \chi^*(a, \phi) \chi(a, \phi) \quad (13)$$

where the integral is over the different matter modes.

An equation for χ may be obtained [17] by multiplying (10) by χ and subtracting it from (8):

$$\begin{aligned} a \psi \left(\hat{H}_M - \langle \hat{H}_M \rangle \right) \chi + \frac{\hbar^2}{M_{pl}^2} (D\psi) \bar{D}\chi &= - \frac{\hbar^2}{2M_{pl}^2} \psi (\bar{D}^2 - \langle \bar{D}^2 \rangle) \chi = \\ &= - \frac{\hbar^2}{2M_{pl}^2} \psi \left[\left(\frac{\partial^2}{\partial a^2} - \langle \frac{\partial^2}{\partial a^2} \rangle \right) - 2 \langle \frac{\partial}{\partial a} \rangle \left(\frac{\partial}{\partial a} - \langle \frac{\partial}{\partial a} \rangle \right) \right] \chi. \end{aligned} \quad (14)$$

We note that the r.h.s. of eqs. (10) and (14) are related to fluctuations, that is they consist of an operator acting on a state minus its expectation value with respect to that state. Henceforth we shall refer to them as RHS fluctuations.

It is convenient to write the above equations in terms of normal rather than covariant (D) derivatives and take into account the contribution from the connections by multiplying the gravity and matter wave functions by a phase. This redefinition leaves invariant the original wave-function Ψ in (9) since the two phases are opposite in sign. With

$$\tilde{\psi} \equiv e^{+i \int^a \mathcal{A} da'} \psi \quad (15)$$

eq. (10) becomes

$$\left(\frac{\hbar^2}{2M_{pl}^2} \frac{\partial^2}{\partial a^2} + a \langle \hat{H}_M \rangle \right) \tilde{\psi} = - \frac{\hbar^2}{2M_{pl}^2} \langle \bar{D}^2 \rangle \tilde{\psi} \quad (16)$$

Further we observe that in the semiclassical (WKB) limit for $\tilde{\psi}$ one has:

$$\tilde{\psi} = W e^{\frac{i}{\hbar} S_{eff}}, \quad (17)$$

where S_{eff} is solution to the Hamilton-Jacobi equation

$$- \frac{1}{2M_{pl}^2} \left(\frac{\partial S_{eff}}{\partial a} \right)^2 + a \langle \hat{H}_M \rangle = 0, \quad (18)$$

and $W = (\partial S_{eff}/\partial a)^{-\frac{1}{2}}$. We analogously define $\tilde{\chi}$ by

$$\tilde{\chi} \equiv e^{-i \int^a \mathcal{A} da'} \chi \quad (19)$$

so that eq. (14) can be written in the form

$$a \tilde{\psi} \left(\hat{H}_M - \langle \hat{H}_M \rangle \right) \tilde{\chi} + \frac{\hbar^2}{M_{pl}^2} \left(\frac{\partial}{\partial a} \tilde{\psi} \right) \frac{\partial}{\partial a} \tilde{\chi} = -\frac{\hbar^2}{2M_{pl}^2} \tilde{\psi} (\bar{D}^2 - \langle \bar{D}^2 \rangle) \tilde{\chi}. \quad (20)$$

On introducing a time derivative

$$i\hbar \frac{\partial}{\partial t} \equiv -i \frac{\hbar}{M_{pl}^2} \frac{\partial S_{eff}}{a \partial a} \frac{\partial}{\partial a} \quad (21)$$

and defining

$$\chi_s \equiv e^{-\frac{i}{\hbar} \int^t \langle \hat{H}_M \rangle dt' - i \int^a da' \mathcal{A}} \tilde{\chi} = e^{-\frac{i}{\hbar} \int^t \langle \hat{H}_M \rangle dt'} \tilde{\chi} \quad (22)$$

eq. (14) may be further rewritten as:

$$\left(\hat{H}_M - i\hbar \frac{\partial}{\partial t} \right) \chi_s = -\frac{\hbar^2}{M_{pl}^2 a} e^{-\frac{i}{\hbar} \int^t \langle \hat{H}_M \rangle dt' - i \int^a da' \mathcal{A}} \left[\frac{\partial \log W}{\partial a} \bar{D} + \frac{1}{2} (\bar{D}^2 - \langle \bar{D}^2 \rangle) \right] \chi_s. \quad (23)$$

which is defined where $\tilde{\psi}$ has support in the semiclassical limit (eq. (17)) and henceforth we shall indicate by a the classical scale factor $a(t)$. We note that on neglecting the rhs of eq. (23) one has a time dependent Schrödinger equation for χ_s . It is important to remember that with this approximation one does not take in account the transitions of the quantum state of the Schrödinger equation caused by the RHS. We shall discuss these transitions in sections 2.4 and 4.

Let us stress that the peculiarities of this coupled system originated from the BO reduction of the WDW equation. The Schrödinger equation, involving time defined in (21), describes the quantum evolution of matter, which in turn leads to the back-reaction appearing in the semiclassical evolution of the scale factor in eq. (18).

2.3 The quantum matter

The quantum evolution of matter, as described by eq. (23), is given by a Hamiltonian:

$$\hat{H}_M = -\frac{\hbar^2}{2a^3} \frac{\partial^2}{\partial \phi^2} + a^3 \frac{m^2}{2} \hat{\phi}^2 \quad (24)$$

which is a harmonic oscillator with a time dependent mass $a^3(t)$ and a constant frequency m and has been previously studied (see for instance [10, 20]) without resorting to the adiabatic approximation [21].

However, since we are interested in a problem for which the evolution of $a(t)$ could have an exponential behaviour, it is much better to employ a technique whereby a time-dependent problem can be solved *exactly* (we work in the Schrödinger representation). This technique is based on an auxiliary *invariant* operator (historically called adiabatic invariant [10]) which satisfies

$$i\hbar \frac{\partial \hat{I}}{\partial t} + [\hat{I}, \hat{H}_M] = 0. \quad (25)$$

The quadratic hermitean invariant originally introduced in [10] has real time-independent eigenvalues and is given by:

$$\hat{I} = \frac{1}{2} \left[\frac{\hat{\phi}^2}{\rho^2} + (\rho \hat{\pi}_\phi - a^3 \dot{\rho} \hat{\phi})^2 \right], \quad (26)$$

where ρ satisfies the following equation:

$$\ddot{\rho} + 3\frac{\dot{a}}{a}\dot{\rho} + m^2\rho = \frac{1}{a^6\rho^3} \quad (27)$$

This invariant \hat{I} is important and useful since a general solution of the time dependent Schrödinger equation with the Hamiltonian (24) can be expressed as:

$$|\chi, t\rangle_s = \sum_n c_n e^{i\varphi_n(t)} |n, t\rangle, \quad (28)$$

where the c_n are time-independent coefficients, $|n, t\rangle$ are the eigenstates of the invariant (26) with real eigenvalues λ_n and the phases $\varphi_n(t)$ are given by

$$\varphi_n(t) = \frac{1}{\hbar} \int_{t_0}^t \langle n, t' | i\hbar \partial_{t'} - \hat{H}_M(t') | n, t' \rangle dt'. \quad (29)$$

The quadratic hermitean invariant (26) can be decomposed in basic linear non-hermitean invariants [20]:

$$\begin{aligned} \hat{I}_b(t) &= e^{i\Theta(t)} \hat{b}(t) = \frac{e^{i\Theta}}{\sqrt{2\hbar}} \left[\frac{\hat{\phi}}{\rho} + i \left(\rho \hat{\pi}_\phi - a^3 \dot{\rho} \hat{\phi} \right) \right], \\ \hat{I}_b^\dagger(t) &= e^{-i\Theta(t)} \hat{b}^\dagger(t) = \frac{e^{-i\Theta}}{\sqrt{2\hbar}} \left[\frac{\hat{\phi}}{\rho} - i \left(\rho \hat{\pi}_\phi - a^3 \dot{\rho} \hat{\phi} \right) \right], \end{aligned} \quad (30)$$

with

$$\Theta(t) = \int_{t_0}^t \frac{dt'}{a^3(t') \rho^2(t')}. \quad (31)$$

The quadratic invariant (26) can then be written as:

$$\hat{I}(t) = \hbar \left(\hat{I}_b^\dagger(t) \hat{I}_b(t) + \frac{1}{2} \right) = \hbar \left(\hat{b}^\dagger(t) \hat{b}(t) + \frac{1}{2} \right); \quad (32)$$

with $[\hat{I}_b, \hat{I}_b^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$ which allows us to identify the pairs of creation and destruction operators, $(\hat{I}_b, \hat{I}_b^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$. The first is used to define a Fock space describing the solutions to the Schrödinger equation, and the second one generates the space of the eigenstates of the invariant defined in (26). As already shown in [22] this difference is important because χ , as defined in (9), belongs to the second space, whereas χ_s , as defined in (22), belongs to the first, of course always on neglecting the RHS fluctuations.

Let us emphasize that the \hat{b} operators that factorize the quadratic invariant are quite different from the creation and destruction operators (which we may denote by \hat{d}) that factorize the Hamiltonian (24):

$$\hat{H}_M = \hbar m \left(\hat{d}^\dagger(t) \hat{d}(t) + \frac{1}{2} \right) \quad (33)$$

with:

$$\begin{aligned} \hat{d} &= \left(\frac{ma^3}{2\hbar} \right)^{\frac{1}{2}} \left(\hat{\phi} + i \frac{\hat{\pi}_\phi}{ma^3} \right), \\ \hat{d}^\dagger &= \left(\frac{ma^3}{2\hbar} \right)^{\frac{1}{2}} \left(\hat{\phi} - i \frac{\hat{\pi}_\phi}{ma^3} \right). \end{aligned} \quad (34)$$

and $[\hat{d}, \hat{d}^\dagger] = 1$. The two sets of d and b operators are related by the following Bogoliubov transformation

$$\begin{aligned} \hat{d} &= B^*(t) \hat{b} + A^*(t) \hat{b}^\dagger \equiv \\ &\equiv \frac{1}{2} \hat{b} \left[\rho(a^3 m)^{\frac{1}{2}} + \frac{1}{\rho(a^3 m)^{\frac{1}{2}}} + i \dot{\rho} \left(\frac{a^3}{m} \right)^{\frac{1}{2}} \right] + \\ &+ \frac{1}{2} \hat{b}^\dagger \left[\rho(a^3 m)^{\frac{1}{2}} - \frac{1}{\rho(a^3 m)^{\frac{1}{2}}} + i \dot{\rho} \left(\frac{a^3}{m} \right)^{\frac{1}{2}} \right], \end{aligned} \quad (35)$$

and similarly for \hat{d}^\dagger .

Since the quantum mechanical problem related to this time-dependent harmonic oscillator can be solved exactly, as in eq. (28), let us first consider the state which best represents the classical behaviour of the scalar field in order to reproduce the previous results [23, 24, 15, 25]. As is known [14] this state is a *coherent* state [13] for \hat{I}_b , which satisfies:

$$\hat{I}_b |\alpha, t\rangle_s = \alpha |\alpha, t\rangle_s \quad (36)$$

where $\alpha \equiv |\alpha| e^{i\beta}$ is a time-independent complex c-number. Such a state satisfies the Schrödinger equation and can also be written as a normalized superposition of eigenstates of the quadratic invariant in (26):

$$|\alpha, t\rangle_s = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, t\rangle_s = e^{-|\alpha|^2/2 - i\Theta/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-in\Theta}}{\sqrt{n!}} |n, t\rangle_b. \quad (37)$$

On defining $\langle \hat{O} \rangle_\alpha \equiv {}_s\langle \alpha, t | \hat{O} | \alpha, t \rangle_s$, the mean value of some physically relevant quantities is:

$$\phi_c \equiv \langle \hat{\phi} \rangle_\alpha = \sqrt{2\hbar} |\alpha| \rho \cos(\Theta - \beta), \quad (38)$$

$$\pi_{\phi,c} \equiv \langle \hat{\pi}_\phi \rangle_\alpha = \sqrt{2\hbar} |\alpha| \left[a^3 \dot{\rho} \cos(\Theta - \beta) - \frac{1}{\rho} \sin(\Theta - \beta) \right] \quad (39)$$

$$\langle (\Delta \hat{\phi})^2 \rangle_\alpha \equiv \langle (\hat{\phi} - \phi_c)^2 \rangle_\alpha = \frac{\hbar}{2} \rho^2$$

$$\langle (\Delta \hat{\pi}_\phi)^2 \rangle_\alpha \equiv \langle (\hat{\pi}_\phi - \pi_{\phi,c})^2 \rangle_\alpha = \frac{\hbar}{2} \left(\frac{1}{\rho^2} + a^6 \dot{\rho}^2 \right), \quad (40)$$

$$\langle (\Delta \hat{\phi})^2 \rangle_\alpha^{\frac{1}{2}} \langle (\Delta \hat{\pi}_\phi)^2 \rangle_\alpha^{\frac{1}{2}} = \frac{\hbar}{2} \sqrt{1 + a^6 \rho^2 \dot{\rho}^2}.$$

We note that the mean value of the field (38) satisfies the classical equation of motion:

$$\ddot{\phi}_c + 3 \frac{\dot{a}}{a} \dot{\phi}_c + m^2 \phi_c = 0, \quad (41)$$

and α does not appear in the uncertainty relation (40), which however increases with time.

The mean value of the matter Hamiltonian for the coherent state is:

$$\begin{aligned} \langle \hat{H}_M \rangle_\alpha &= \frac{1}{2a^3} \left[\langle \hat{\pi}_\phi \rangle_\alpha^2 + m^2 a^6 \langle \hat{\phi} \rangle_\alpha^2 + \frac{\hbar}{2} \left(\dot{\rho}^2 a^6 + \frac{1}{\rho^2} + m^2 \rho^2 a^6 \right) \right] \\ &= H_{cl} + H_0 \equiv a^3 (\mu_{cl} + \mu_0) \end{aligned} \quad (42)$$

where H_{cl} is the classical Hamiltonian for the scalar field ϕ_c defined in eq.(38), H_0 is the zero-point Hamiltonian and μ_{cl} , μ_0 are respectively the classical and zero point energy-densities. All these quantities are time-dependent. In order to consider the above semiclassical field as a fluid we further define, in analogy to the classical case, a *pressure* operator:

$$\hat{p} = \frac{1}{2a^3} \left(\frac{\hat{\pi}_\phi^2}{a^3} - m^2 a^3 \hat{\phi}^2 \right). \quad (43)$$

Its expectation value in a coherent state is:

$$\langle \hat{p} \rangle_\alpha = \frac{1}{2a^3} \left[\frac{\pi_{\phi,c}^2}{a^3} - m^2 a^3 \phi_c^2 + \frac{\hbar}{2} \left(\dot{\rho}^2 a^3 + \frac{1}{a^3 \rho^2} - m^2 \rho^2 a^3 \right) \right] \equiv p_{cl} + p_0 \quad (44)$$

and we have the usual fluid conservation law:

$$\dot{\mu}_{cl} + \dot{\mu}_0 + 3 \frac{\dot{a}}{a} (\mu_{cl} + \mu_0 + p_{cl} + p_0) = 0. \quad (45)$$

2.4 Higher order corrections to semiclassical theory

We are interested in the corrections given by the rhs of eq. (10) for gravity and by the rhs of eq. (14) for χ in a coherent state or an n-particle state [26], for which $\langle \chi | \chi \rangle = 1$. For this purpose we note that the χ wave function introduced in (9) can be constructed by using the b, b^\dagger operators. Let us then indicate by $|\alpha\rangle_b$ the coherent state associated with the operator \hat{b} while the $|n\rangle_b$ are the eigenstates of the associated number operator. Further, we shall derive expressions valid during the inflationary regime ($a \gg 1$), so that some simplification will occur.

Let us first consider the corrections to the gravity equation. With the introduction of the semiclassical time in eq. (21), we can write:

$$\langle \chi | \bar{D}^2 | \chi \rangle = \frac{1}{\dot{a}^2} \left(\left| \frac{\partial}{\partial t} | \chi \rangle_b \right|^2 - |b \langle \chi | \frac{\partial}{\partial t} | \chi \rangle_b|^2 \right) = \frac{1}{\dot{a}^2} P_\chi. \quad (46)$$

On defining the Hubble parameter as $H = \frac{\dot{a}}{a}$, the semiclassical version of eq. (16) leads to:

$$H^2 = \frac{8\pi G}{3a^3} \langle \chi | \hat{H}_M | \chi \rangle + \frac{\hbar^2}{a^4 M_{pl}^4} \langle \chi | \bar{D}^2 | \chi \rangle = \frac{8\pi G}{3a^3} \langle \hat{H}_M \rangle + \frac{\hbar^2}{a^6 M_{pl}^4 H^2} P_\chi. \quad (47)$$

With the help of formulae in the appendix one obtains:

$$P_n = \left| \frac{\partial}{\partial t} | n \rangle_b \right|^2 - |b \langle n | \frac{\partial}{\partial t} | n \rangle_b|^2 = 2m^2 |A|^2 |B|^2 (n^2 + n + 1) \quad (48)$$

$$\begin{aligned} P_\alpha = \left| \frac{\partial}{\partial t} | \alpha \rangle_b \right|^2 - |b \langle \alpha | \frac{\partial}{\partial t} | \alpha \rangle_b|^2 &= \left[\dot{\Theta} - m(1 + 2|A|^2) \right]^2 |\alpha|^2 + 2m^2 |A|^2 |B|^2 (1 + 2|\alpha|^2) + \\ &+ 2m (\alpha^2 AB^* + \alpha^{*2} BA^*) \left(m(1 + 2|A|^2) - \dot{\Theta} \right). \end{aligned} \quad (49)$$

We note that for the coherent case the terms in α of order higher than quadratic cancel. For completeness we also write the average matter Hamiltonian for an $|n\rangle_b$ state:

$${}_b \langle n | \hat{H}_M | n \rangle_b = \hbar m (|A|^2 + \frac{1}{2}) (2n + 1) = a^3 \mu_0 (2n + 1) \quad (50)$$

and on solving eq. (47) for H^2 one obtains

$$H^2 = \frac{2 \langle \hat{H}_M \rangle}{M_{pl}^2 a^3} \left(\frac{1 + \sqrt{1 + \frac{\hbar^2 P_\chi}{\langle \hat{H}_M \rangle^2}}}{2} \right) = \frac{2 \langle \hat{H}_M \rangle}{M_{pl}^2 a^3} \xi \quad (51)$$

Let us now estimate the corrections, for $a \gg 1$, to the results obtained when the rhs is neglected for the coherent, the $|n\rangle_b$ and the vacuum states respectively. One first observes that the time dependence of the rhs in this limit is the same as that of the Hamiltonian, so that their ratio becomes time independent. For the coherent case in the $|\alpha| \rightarrow \infty$ limit we have:

$$\frac{\hbar^2 P_\alpha}{{}_b \langle \alpha | \hat{H}_M | \alpha \rangle_b^2} \approx \frac{2 + \cos 2\beta}{4|\alpha|^2} \quad (52)$$

which goes to zero, so that one may expect no correction in the classical limit. For the $|n\rangle_b$ state case, in the $n \rightarrow \infty$ limit, we have

$$\frac{\hbar^2 P_n}{{}_b \langle n | \hat{H}_M | n \rangle_b^2} \approx \frac{1}{2} \quad (53)$$

which leads to an 11% correction. Finally for the vacuum case ($\alpha = n = 0$) we obtain:

$$\frac{\hbar^2 P_0}{{}_b \langle 0 | \hat{H}_M | 0 \rangle_b^2} \approx 2 \quad (54)$$

which gives a correction $(\xi - 1)$ of order 36%.

We now consider the matter equation. It is convenient to introduce

$$|\phi_{\chi_2, \chi_1}\rangle = |\chi_2\rangle - \langle\chi_1|\chi_2\rangle|\chi_1\rangle. \quad (55)$$

As the corrections are an operator we shall examine their matrix elements with respect to the $|n\rangle_b$ states and the coherent states (an overcomplete non-orthogonal set). The general form will be

$$\begin{aligned} \frac{\hbar^2}{2M_{pl}^2 a \dot{a}^2} & \left[-2 \frac{\partial W}{\partial t} \langle\phi_{\chi_2, \chi} | \frac{\partial}{\partial t} | \chi\rangle - \dot{a}^2 \frac{\partial}{\partial a} (\langle\phi_{\chi_2, \chi} | \frac{\partial}{\partial a} | \chi\rangle) + \right. \\ & \left. \langle\phi_{\chi_2, \chi} | \frac{\partial}{\partial t} \frac{\partial}{\partial t} | \chi\rangle + 2 \langle\chi | \frac{\partial}{\partial t} | \chi\rangle \langle\phi_{\chi_2, \chi} | \frac{\partial}{\partial t} | \chi\rangle \right] \end{aligned} \quad (56)$$

In the large a limit, during inflation, the first two terms of eq. (56) will be subdominant with respect the last two by a factor of order a^3 , hence we may neglect them and denote the last two by $Q_{\chi_2, \chi}$.

Using the results given in the appendix and in eqs. (50) and (56) one has for the ratio of the the matrix elements between the $|n\rangle_b$ and $|m\rangle_b$ states and the average Hamiltonian :

$$\begin{aligned} \frac{Q_{m,n}}{b\langle n | \hat{H}_M | n \rangle_b} &= \frac{1}{4\xi(2n+1)^2} \left[-4\sqrt{n(n-1)}\delta_{m,n-2} + \sqrt{n(n-1)(n-2)(n-3)}\delta_{m,n-4} + \right. \\ & \left. \sqrt{(n+1)(n+2)(n+3)(n+4)}\delta_{m,n+4} + 4\sqrt{(n+1)(n+2)}\delta_{m,n+2} \right] \end{aligned} \quad (57)$$

which are of order unity for the vacuum ($n = 0$) state and one order smaller, but finite, for large n . The form obtained reveals that the allowed transitions will eventually lead the system to spread over a superposition of states

In the same manner we obtain expressions for the coherent states in the large a limit. Only the last two terms of eq. (56) contribute giving

$$\begin{aligned} Q_{\beta, \alpha} &= \frac{b\langle\beta|\alpha\rangle_b \hbar^2 m^2 |A|^4}{2M_{pl}^2 a^3 H^2} (\beta^* - \alpha^*) \left[(\alpha^* + \beta^* + 4\alpha)(4 + \beta^{*2}) - \right. \\ & \left. \alpha^*(\alpha^{*2} + \beta^{*2}) + \alpha^2(8\alpha + 11\alpha^* + 3\beta^*) + 4\alpha\alpha^{*2} \right] \end{aligned} \quad (58)$$

In order to see how large such corrections are in the $|\alpha| \rightarrow \infty$ limit, when the system is characterized by a behaviour close to the classical, we set $\gamma \equiv \beta - \alpha$ and obtain

$$\begin{aligned} \frac{Q_{\beta, \alpha}}{b\langle\alpha|\hat{H}_M|\alpha\rangle_b} &= \frac{e^{-|\gamma|^2/2 + i \text{Im}(\alpha\beta^*)} \hbar^2 m^2 |A|^4}{4\xi b\langle\alpha|\hat{H}_M|\alpha\rangle_b^2} \gamma^* \left[8\alpha^* + 16\alpha + 4\gamma^* + 3(\alpha^2 + \alpha^{*2} + \gamma^*\alpha^* + \right. \\ & \left. \gamma^{*3} + 8\alpha\alpha^{*2} + 8|\alpha|^2\gamma^* + 4\alpha\gamma^{*2} + 8\alpha^3 + 14\alpha^*\alpha^2) \right] \underset{|\alpha| \gg 1}{\sim} \mathcal{O}\left(\frac{1}{|\alpha|}\right). \end{aligned} \quad (59)$$

We note that the above quantity is bounded in γ so that in the $|\alpha| \rightarrow \infty$ limit every transition away from a coherent state is suppressed and thus, as expected, one recovers the classical behaviour of the system. Further the behaviour $\mathcal{O}(\frac{1}{|\alpha|})$ is also valid for any a .

Let us finally exhibit the ratio for the vacuum case ($\alpha = 0$):

$$\frac{Q_{\beta, 0}}{b\langle 0 | \hat{H}_M | 0 \rangle_b} = \frac{b\langle\beta|0\rangle_b \hbar^2 m^2 |A|^4}{4\xi b\langle 0 | \hat{H}_M | 0 \rangle_b^2} (\beta^{*4} + 4\beta^{*2}) = \frac{1}{4\xi} (\beta^{*4} + 4\beta^{*2}) e^{-|\beta|^2/2}. \quad (60)$$

Of course if one multiplies the above by $b\langle n|\beta\rangle_b$ and integrates $\int d\mu(\beta)$, where the measure ($d\mu(\beta) = \frac{d\text{Re}\beta d\text{Im}\beta}{\pi}$) is such that the completeness relation is $\int d\mu(\beta) |\beta\rangle_b \langle\beta| = 1$, we re-obtain the matrix elements for $n = 0$ in eq. (57) as expected.

Since we are also interested in the time evolution for the system initially in the vacuum state we shall also estimate in section 4 the transition rates (to the $|2\rangle_b$ and $|4\rangle_b$ states) which are generated by the correction terms on the rhs of the matter equation.

3 The analysis of the dynamical systems

In this section, in preparation for the numerical analysis, we first derive an effective equation describing the evolution of semiclassical gravity after “averaging” over the quantum matter degree of freedom. In our model, on neglecting the RHS fluctuations, this equation will not depend on the matter quantum state. Subsequently we shall study the dynamics of the matter quantum state in order to understand and possibly evaluate the effect of the approximations made. In particular we shall consider in detail the evolution for a coherent state, a vacuum state and a thermal state.

3.1 Effective equation for gravitation.

Let us begin by considering matter in a generic quantum state $|\chi\rangle_s$ satisfying the Schrödinger equation with Hamiltonian \hat{H}_M . The backreaction on gravity is described by the equation

$$H^2 = \frac{8\pi G}{3a^3} {}_s\langle \chi | \hat{H}_M | \chi \rangle_s = \frac{8\pi G}{3} {}_s\langle \chi | \hat{\mu} | \chi \rangle_s \equiv \frac{8\pi G}{3} \mu \quad (61)$$

where $\hat{\mu} = \hat{H}_M/a^3$ is the energy density operator and μ its average. On differentiating equation (61) with respect to time one obtains:

$$\dot{H} + \frac{3}{2}H^2 = \frac{4\pi G}{3a^3} {}_s\langle \chi | \frac{1}{H} \frac{\partial \hat{H}_M}{\partial t} | \chi \rangle_s = -4\pi G {}_s\langle \chi | \hat{p} | \chi \rangle_s \equiv -4\pi G p \quad (62)$$

where \hat{p} is the pressure operator defined in (43) and p its average. One notes that $\dot{H} = -4\pi G(p + \mu)$ so that a fluid-like equation is satisfied:

$$\dot{\mu} + 3H(p + \mu) = 0 \quad (63)$$

We now differentiate eq. (62) with respect to time and use the relations (61) and (62) in order to eliminate the average, with respect to $|\chi\rangle_s$, of the Hamiltonian and of its time derivative. One then obtains

$$\ddot{H} + 6H\dot{H} = \frac{4\pi G}{a^3} m^2 {}_s\langle \chi | \{\hat{\phi}, \hat{\pi}_\phi\} | \chi \rangle_s \quad (64)$$

where the term on the RHS comes from the commutator $[\hat{p}, \hat{H}_M]$. Further we perform yet another time derivative of (64) obtaining

$$\ddot{H} + 6\dot{H}^2 + 6H\ddot{H} = -\frac{12\pi G}{a^3} m^2 H {}_s\langle \chi | \{\hat{\phi}, \hat{\pi}_\phi\} | \chi \rangle_s + \frac{4i\pi G}{\hbar a^3} m^2 {}_s\langle \chi | [\hat{H}_M, \{\hat{\phi}, \hat{\pi}_\phi\}] | \chi \rangle_s \quad (65)$$

One easily computes the commutator in the last term $[\hat{H}_M, \{\hat{\phi}, \hat{\pi}_\phi\}] = -4i\hbar a^3 \hat{p}$, then, on using the relations (62) and (64), one finally obtains an equation for H only:

$$\ddot{H} + 6\dot{H}^2 + 4\dot{H} + 9\ddot{H}H + 18H^2(\dot{H} + \frac{1}{3}) = 0, \quad (66)$$

where we have scaled the time by m^{-1} and henceforth we shall use this new scaled time.

The above equation is valid for every quantum state, in particular for a coherent state, and, on taking the classical limit for matter, also for the classical field. Thus the classical model of chaotic inflation can be described by this equation. Indeed, it is known that the classical system is governed by the system of equations

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \left(\frac{\phi_c^2 + \dot{\phi}_c^2}{2} \right) \\ \ddot{\phi}_c + 3\dot{\phi}_c \sqrt{\frac{8\pi G}{3} \left(\frac{\phi_c^2 + \dot{\phi}_c^2}{2} \right)} + \phi_c &= 0 \end{aligned} \quad (67)$$

which gives a non linear second order differential equation in H but, as is easy to check, also implies eq. (66). Hence the classical system has a dynamics which lives in a two-dimensional surface in the (H, \dot{H}, \ddot{H}) phase space. This surface is an invariant manifold

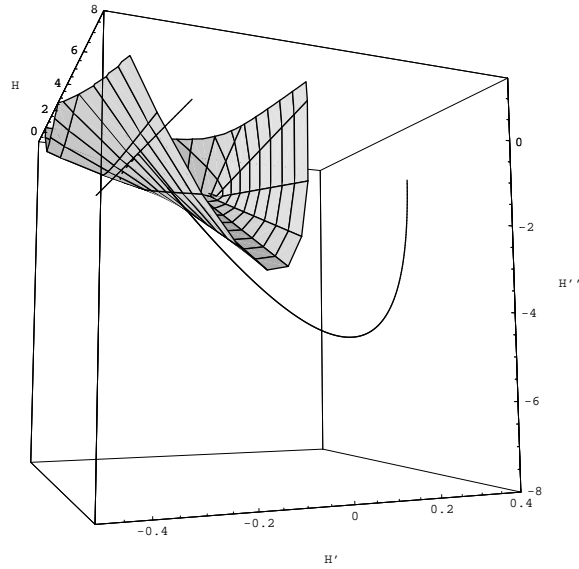


Figure 1: In gray we have the invariant manifold corresponding to the classical dynamics (67). Also exhibited are a trajectory related to a highly non-classical state with initial conditions $(H = 4, \dot{H} = 0.4, \ddot{H} = 0)$ and a line, given by $(\dot{H} = -1/3, \ddot{H} = 0)$ and not intersecting the invariant surface, which is approached by the system in the inflationary phase both for classical and quantum matter; after reaching it the trajectory will lie close to the invariant surface. Let us note that in the figures (1 and 2) we use a prime to denote a time derivative.

under the dynamics of eq. (66) which was obtained for quantum matter. It is the initial conditions that distinguish the two cases: in the classical case there is a precise relation between $H(t_0), \dot{H}(t_0)$ and $\ddot{H}(t_0)$ which is not satisfied in the quantum case as can easily be seen, for example, for the vacuum or a coherent state. The trajectories for a generic quantum state do not lie on this surface, but they approach it asymptotically during evolution, however they will never exactly lie on it (see Fig. 1).

In order to understand how the inflationary regime can be reached one may study the set of initial conditions in the three dimensional (H, \dot{H}, \ddot{H}) phase space which lead to it. Let us note that the only fixed point in the finite region for the vector field associated with eq. (66) is given by $(H, \dot{H}, \ddot{H}) = (0, 0, 0)$. This point is asymptotically stable and the divergence of the vector field is negative everywhere for positive H , so that every trajectory will fall towards it.

On the basis of the numerous analyses already performed on the classical system [15, 23, 24, 25] one expects the value of H to change during the eventual inflationary phase, decreasing (on starting from initial conditions on the two-dimensional invariant manifold shown on Fig. 1) from values generally much greater than one in the units we use (Planck mass). This implies, on comparing the magnitude of the different terms of the eq. (66), that one can expect, from the last two terms, unless H is small,

$$\ddot{H} \approx 0 \quad , \quad \dot{H} \approx -\frac{1}{3}. \quad (68)$$

Let us note that this is the same behaviour as the classical case, and is quite different from a true de Sitter evolution for which $H = \text{const}$. We remark that this is the same behaviour as is also found by the authors of [15], who describe the curve corresponding to (68) by $\dot{\phi}_c \approx \text{const}$. Therefore in the three dimensional phase space, the trajectories starting from different points (which are related to the initial conditions), for both the classical and quantum matter cases will, if leading to an inflationary regime and after an eventually transient phase, converge to a region around the curve characterized by (68) and a decreasing H . This is what is shown

and commented in Fig. 1. Since one is also interested in the pressure/energy-density ratio, let us write the expression to be used later in the numerical simulations:

$$\frac{p}{\mu} = -1 - \frac{2\dot{H}}{3H^2}. \quad (69)$$

remembering that it is valid for any quantum state satisfying the Schrödinger equation.

The above approach for gravitation is general and does not require any direct solution of the Schrödinger equation for matter. However it is important to know the matter state, and for this reason we introduced the technique of the invariants. On using them it is possible to construct coherent states to study the classical limit and estimate if a quantum state, satisfying the Schrödinger equation, defined by the Hamiltonian \hat{H}_M will be modified by RHS fluctuations, possibly before the end of inflation. We shall attempt perturbative estimates in section 4.

3.2 Matter states.

For the coherent state one has a four dimensional dynamical system, which is given by eq. (27) for ρ and:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G(\mu_{cl} + \mu_0) \quad (70)$$

$$\dot{\Theta} = \frac{1}{a^3\rho^2} \quad (71)$$

We are interested in evolution from an initial state which is well classified in terms of the Hamiltonian at the initial time ($t = t_0$), that is a coherent state constructed from the particle annihilation and creation operators which factorize the Hamiltonian. This choice corresponds to requiring the invariant and the Hamiltonian coincide at $t = t_0$ and the Bogoliubov coefficients will be $A(t_0) = 0$ and $B(t_0) = 1$, leading to the conditions:

$$\begin{aligned} \rho^2(t_0) &= \frac{1}{ma^3(t_0)} \\ \dot{\rho}(t_0) &= 0 \end{aligned} \quad (72)$$

Subsequently we shall consider a larger set of initial states not satisfying the above. It is remarkable that the above requirement relates the fluctuations of the scalar field given by eq. (40) to the scale factor of the universe. Indeed at the quantum level the scale factor a appears inextricably in the equations (already in the energy density, see for example eq. (42)), this is the basic reason for which the semiclassical system is three dimensional. From eq. (72), corresponding to a possible choice of initial conditions, it is clear that the smaller is the scale factor the larger is ρ , which would actually be infinite if a were zero as is classically allowed.

Let us now examine the vacuum state. This state is interesting even if it is obvious that it will be the first to be changed by the transitions generated by the RHS fluctuations of the matter equation (23). The vacuum case corresponds to $\alpha = 0$ ($\mu_{cl} = 0$ in eq. (70)), so that Θ decouples and we may eliminate a in eq. (27) obtaining an equation for ρ and H . In analogy with eq. (62), on scaling ρ by $\sqrt{G\hbar}$, one also obtains:

$$\ddot{\rho} + 3H\dot{\rho} + \frac{\dot{\rho}^2 - \frac{3}{2\pi}H^2}{\rho} + 2\rho = 0 \quad (73)$$

$$\dot{H} + 3H^2 - 2\pi\rho^2 = 0. \quad (74)$$

In this case we study the evolution of an initial state annihilated by the invariant destruction operator which is not necessarily the Hamiltonian one, unless the conditions given in (72) are imposed, and parametrize it in terms of the initial conditions ($\rho(t_0)$, $\dot{\rho}(t_0)$ and $H(t_0)$) or ($H(t_0)$, $\dot{H}(t_0)$ and $\ddot{H}(t_0)$). We observe that if the scalar field is in the n eigenstate of the invariant, the dynamical system is given by eq. (73) with the term H^2 divided by $(2n+1)$ and by eq. (74) with the term ρ^2 multiplied by $(2n+1)$.

We may now consider a thermal (not pure) initial state which is also characterized by zero mean values of the matter field and of its derivative. Such a state could arise within the context of our BO approach on introducing a complex, rather than real, time, thus generalizing our previous considerations. In order to describe such an initial state we consider a density matrix constructed using as basis a given number of quanta of the Hamiltonian (at $t = t_0$). If at time t_0 the conditions given in (72) are satisfied, then the time evolution of the basis will be given exactly by the eigenstates of the quadratic invariant so that the density matrix will be

$$\hat{\rho} = (1 - e^{-\beta m}) \sum_k e^{-\beta m k} |k, t\rangle \langle k, t| \quad (75)$$

where $\beta = 1/k_B T$ and of course during evolution it will no longer be a thermal state of the Hamiltonian. The operator $\hat{\rho}$ satisfies the Liouville equation so that also for this state one easily derives the effective equation (66). In fact using the known evolution equation for the $\hat{\rho}$ operator we can compute, on taking traces, the mean value of all interesting operators. In particular one has for the average matter Hamiltonian density:

$$Tr \left(\hat{\rho} \frac{\mathcal{H}_M}{a^3} \right) = \mu_0 \left(1 + \frac{2}{e^{\beta m} - 1} \right) \quad (76)$$

and for the matter fluctuations

$$Tr(\hat{\rho} \hat{\phi}^2) = \frac{\hbar}{2} \rho^2 \left(1 + \frac{2}{e^{\beta m} - 1} \right). \quad (77)$$

These differ from the vacuum case by an extra factor which increases with the temperature, as expected. Thus, even for not so large values of ρ we could still have large fluctuations of the matter field if the temperature is high.

4 Numerical Simulations.

In this section we exhibit the results of the numerical analyses of the previously described dynamical situations.

Let us first consider the basic effective equation (66), which is valid for every quantum system within the framework of the approximations made. We find that, starting from initial conditions very far from the “classical” surface, the inflationary regime coincides with the region described by eq. (68) and the trajectory will lie close to the two-dimensional invariant surface previously mentioned. This is illustrated in Fig. 1.

In order to have a better idea of the different possible trajectories arising from differing initial conditions we show in Fig. 2 a projection of the trajectories on the H - \dot{H} plane. From the picture it is evident that trajectories will lie, to within a very good approximation, for some time on the curve given by (68). Therefore it is evident that we have a large set of initial conditions which will lead to an inflationary stage. One also sees that the number of e-folds reached at the end of the inflationary stage depends on the value of H at the moment when each trajectory approaches this curve. In fact it can be seen that during the inflationary stage starting at $t = t_i$ one has

$$\frac{a(t)}{a(t_i)} \approx \exp \left(H(t_i) t + \dot{H}(t_i) \frac{t^2}{2} \right) \quad (78)$$

and thus the total number of e-folds is

$$N = \int_{t_i}^{t_f} H(t) dt \approx - \frac{H^2(t_i)}{2\dot{H}(t_i)} \quad (79)$$

where t_f denotes the time at which inflation stops and we have considered $H(t_f) \ll H(t_i)$.

In order to compare our results with the already known behaviour of classical models [15, 23, 24, 25] we shall start from a system described by a coherent state which will be squeezed during time evolution. The results of the analysis are easy to understand from the

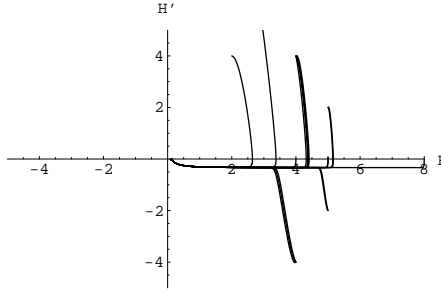


Figure 2: Trajectories with different initial conditions projected on the H - \dot{H} plane. The bold lines reflect a superpositions of curves having quite different \dot{H} for some time. It is evident that all the curves reach the common region described by eq. (68).

structure of equation (70) on noting that H (or a) is completely governed by the matter Hamiltonian density which has two contributions, one from the vacuum and another which is exactly that expected from classical dynamics.

Therefore if we are in a coherent state characterized by an average number of quanta $|\alpha|^2 \gg 1$, the vacuum contribution will be negligible and the resulting evolution will be close to the one given by the classical homogeneous matter field. For such a case we show in Fig. 3 the e-fold number, the Hubble parameter, the mean field fluctuation, the pressure/energy-density ratio and the average field $\langle \alpha | \hat{\phi} | \alpha \rangle$, which corresponds to the classical quantity ϕ_c , resulting from the dynamics. We start from a condition, known to lead to inflation, with the average value of the field starting the slow rollover phase; more precisely, working in units of the Planck mass, we have taken for the inverse Compton wavelength $m = 10^{-6}$, imposed initial condition (72) and $|\alpha| = 2.334 \cdot 10^3$, $\beta = 0$, $\rho(t_0) = 10^{-3}$, $\dot{\rho}(t_0) = 0$ so that $\phi(t_0) = 3.3$, $\dot{\phi}(t_0) = 0$. The resulting initial conditions for equation (66) are $H = 6.7539$, $\dot{H} = -6.283 \cdot 10^{-6}$ and $\ddot{H} = 6.878 \cdot 10^{-15}$ (all these numerical values are for the rescaled dimensionless quantities and time scaled by m^{-1}). After a short transient, a linear decrease of H and ρ until the inflation period ends (with an e-fold number of 70) is evident. The behaviour of the ratio between the pressure and the energy density remains, as expected, close to a value of -1 for most of the inflationary period and later starts to oscillate.

From the point of view of the matter state, it is interesting to note that the role played in the classical case by the value of the field is now replaced by ρ , which is related to the quantum fluctuations of the homogeneous field. During inflation ρ decreases linearly and after this “dissipation” it will start to oscillate above zero. We also note that the average value of the field behaves as the corresponding classical quantity, decreasing during inflation and oscillating after it (in the figure we use a logarithmic scale which covers only positive values).

Let us consider the other extreme case: that of the invariant vacuum (which eventually can also be the Hamiltonian vacuum at $t = t_0$). For this state the average field and time derivative of the field are zero. On neglecting the transitions induced by the RHS of the matter equation, it is clear that for high enough fluctuations (large values of ρ) the system could undergo an inflationary phase. In such a case the behaviour of a , H , ρ , p/μ is the same as that of the coherent case plotted in Fig. 3. However the presence of a non negligible term on RHS implies that the Schrödinger equation will be a rough approximation to the real matter equation and we are then unable to predict the evolution of the system.

Since the vacuum is an extreme case we feel that it is interesting to try to further analyze such a situation. Let us then estimate perturbatively the time necessary for the corrections to produce a large transition rate such that it would be meaningless to consider the normal (Schrödinger) evolution for the vacuum. After such a time we shall no longer be able to

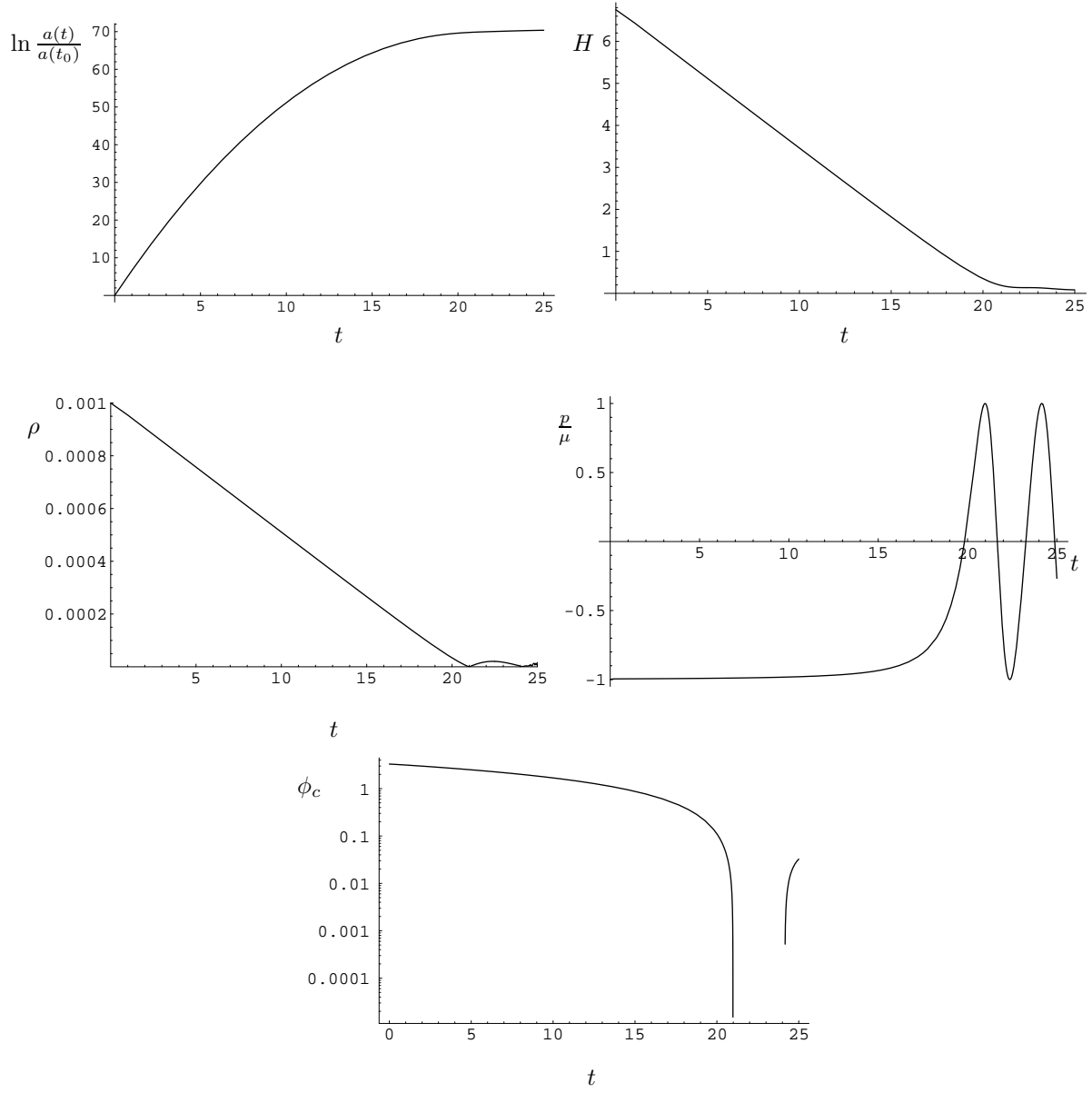


Figure 3: For matter in a coherent state we show the time behaviour of the e-fold number, the Hubble parameter H , the mean value ρ for the fluctuations of the field, the pressure/energy-density ratio and of the average value of the field $\phi_c = \langle \phi \rangle_\alpha$.

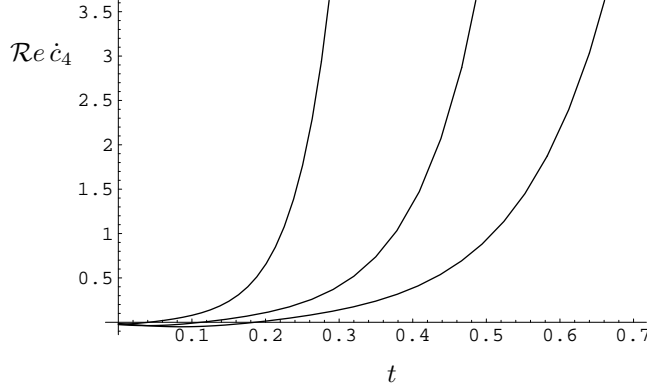


Figure 4: For an inflationary region with different values for ρ (for H) which would lead, in absence of RHS correction, to an inflation with e-fold number of 70, 25, 15, the corresponding (from left to right) time behaviour of the transition rate eq. (81) is shown. For the above the transition rate becomes significant and possibly modifies the inflationary regime after 1%, 4% and 6% respectively of the inflationary period in the absence of RHS fluctuation corrections.

make any prediction and if the system was, or was about to enter, in an inflationary stage it may or may not thus continue. We rewrite the full matter equation (with the RHS) as

$$\left(i\hbar \frac{\partial}{\partial t} - \hat{H}_M\right) |\chi\rangle_s = -e^{i\varphi} \hat{F}_\chi |\chi\rangle \quad (80)$$

where φ and \hat{F}_χ are defined by comparison with eqs. (22) and (23). The full higher order non-linear equation is clearly too complicated to solve. Hence we estimate the transitions to other states which are solutions of the unperturbed Schrödinger equation, as is customarily done in perturbation theory. This means we expand $|\chi\rangle_s = \sum_n c_n(t) |n\rangle_s$ and evaluate the contributions of the perturbation to lowest order [27], thus we consider the operator \hat{F}_0 which depends on the $|0\rangle$ state. One then obtains for the transition rate:

$$-i\hbar \frac{\partial c_n}{\partial t} = e^{in\Theta} \langle n | \hat{F}_0 | 0 \rangle \quad (81)$$

Further we must choose the initial values for ρ , $\dot{\rho}$ and a . The following relations, obtained from eqs. (42) and (70) are useful for this purpose:

$$H = \sqrt{\frac{2\pi G\hbar}{3}} \sqrt{\rho^2 - \frac{\dot{H}}{2\pi G\hbar}}, \quad \dot{H} = -2\pi G\hbar \left(\dot{\rho}^2 + \frac{1}{m^2 a^6 \rho^2} \right), \quad \ddot{H} = -6H\dot{H} + 4\pi G\hbar \rho \dot{\rho} \quad (82)$$

We first consider the case for which the state is initially in an inflationary regime so that the relations in (68) are satisfied. On requiring a certain number of e-folds we find the value of H needed and hence the necessary ρ and $\dot{\rho}$ and a . For example values $H > 1$ will lead to $\rho \approx 0.7H$ and $\dot{\rho} \approx -0.23$ in the chosen units. On using these values we have computed the quantity in eq. (81). In particular in Fig. 4 we show the real part of \dot{c}_4 for three different cases leading to an e-fold of 70, 25 and 15 if the corrections were not present (the right hand side of Eq. (81) has comparable real and imaginary parts, for this reason the contribution to the real part of \dot{c}_4 , plotted in Fig. 4, due to a possible rapidly oscillating phase of c_4 is excluded). We note that for smaller initial H (which would lead to less expansion) the transition rate becomes significant at a later time. If, on the other hand, we consider an initial vacuum satisfying the conditions of eq. (72) (as yet we are not in an inflationary regime), the transition rate is non-negligible from the beginning and as a consequence we simply cannot use the Schrödinger equation to evolve it (analogous considerations hold for c_2).

Little can be added on the evolution from a thermal state, on neglecting the RHS, since it is similar the vacuum case. The basic difference is that the thermal factor in eq. (76) will increase, depending on the temperature, the energy density.

Let us end this section by adding a brief comment about *reheating*. It is known that an inflationary stage during which the scale factor is accelerated should be associated with a massive entropy production in order to solve the problems of the standard cosmology [28]. However, also for states of the matter field with a vanishing mean value persisting till the end of an inflationary phase, ρ would oscillate with the same period the classical field would have had, but its amplitude would decrease in a more complicated way (see, for instance, eq. (73) for the vacuum case). The similarity of the behaviour of scalar field fluctuations for classical and vacuum states has already been noted [29] in an exact de Sitter space-time for long wavelength modes. For these reasons in a model where a second field χ is coupled through a $\phi^2\chi^2$ term to the inflaton, the vanishing value of the scalar field does not seem to prevent the possibility of a reheating process after inflation. In any case we realize that the inclusion of all the other modes is necessary to tackle this program [30].

5 Conclusions

In this paper we have studied a massive homogeneous minimally coupled scalar field in a flat RW metric. After having quantized the system we recovered, through a BO decomposition, the semiclassical evolution for gravity and a Schrödinger equation for matter, always on neglecting RHS fluctuation corrections. Such a treatment is however an approximation which could break down for a matter quantum state because of the growth of the corrections. Since we do not know how to predict exactly the dynamical consequences of the RHS fluctuations, we evaluated them perturbatively and we studied in detail the properties of the dynamical system and the time scale at which the corrections are no longer negligible.

In particular we found:

(1) The dynamical semiclassical gravity system for a generic quantum state of the scalar field is described by a non-linear third order differential equation for the Hubble parameter. We have seen that the classical system lives in an invariant surface of the three dimensional phase space and that an inflationary regime (with $\dot{H} \simeq -\frac{1}{3}$) is experienced for a generic quantum state of the scalar field.

(2) The time scale at which the corrections are no longer negligible depends on the "classicality" of the quantum state for the scalar field. In the classical limit ($|\alpha|^2 \rightarrow +\infty$) these corrections are suppressed while for a generic quantum state they are no longer negligible before inflation ends. For the vacuum state the corrections could become important, in the least favorable case (for example by choosing particular initial conditions), at the same time scale as that for which inflation starts.

In any case we feel that from this analysis it emerges that inflation can arise from a quantum mechanical initial matter state, or at least a coherent one, and not just a classical state. Such a situation was envisaged in the earlier pioneering works [1, 31].

Let us conclude by again noting that the main problems encountered in considering inflation from a non-classical state of the scalar field are related to the difficulties in solving the full matter equation (80). One would like to at least evaluate the effects of the RHS fluctuations non-perturbatively and see how an inflationary regime evolves from a generic quantum matter state. We intend to investigate this and other points in more detail, as well as applications of the above method to other physical problems [32].

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7 Appendix

We here compute the time derivatives of the $|n\rangle_b$ and of the coherent states for the destruction operator b satisfying

$$\frac{\partial \hat{b}}{\partial t} = \frac{i}{\hbar} [\hat{b}, \hat{H}_M] - i\dot{\Theta}\hat{b}. \quad (83)$$

The coherent state $|\alpha\rangle_b$ is defined by applying the operator \hat{D}

$$\hat{D} = e^{\alpha\hat{b}^\dagger - \alpha^*\hat{b}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{b}^\dagger} e^{-\alpha^*\hat{b}} \quad (84)$$

to the vacuum state $|0\rangle_b$, for which $\hat{b}|0\rangle_b = 0$. Therefore

$$\frac{\partial |\alpha\rangle_b}{\partial t} = \frac{\partial \hat{D}}{\partial t} |0\rangle_b + \hat{D} \frac{\partial |0\rangle_b}{\partial t} = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\partial}{\partial t} |n\rangle_b \quad (85)$$

with

$$\frac{\partial}{\partial t} |n\rangle_b = \left[\frac{\partial}{\partial t} \hat{b}^{\dagger n} + \hat{b}^{\dagger n} \frac{\partial}{\partial t} \right] |0\rangle_b \quad (86)$$

where, on recalling the equations satisfied by χ , we have for the second term in (86)

$$\begin{aligned} \frac{\partial |0\rangle_b}{\partial t} &= \left({}_b\langle 0 | \frac{\partial}{\partial t} + \frac{i}{\hbar} \hat{H}_M | 0 \rangle_b \right) |0\rangle_b - \frac{i}{\hbar} \hat{H}_M |0\rangle_b = \\ &= {}_b\langle 0 | \frac{\partial}{\partial t} | 0 \rangle_b - imBA^*\hat{b}^{\dagger 2} | 0 \rangle_b. \end{aligned} \quad (87)$$

On using

$$\frac{\partial \hat{b}^\dagger}{\partial t} = \frac{i}{\hbar} [\hat{b}^\dagger, \hat{H}_M] + i\dot{\Theta}\hat{b}^\dagger. \quad (88)$$

and

$$[\hat{b}^\dagger, \hat{H}_M] = \hbar m [-(1+2|A|^2)b^\dagger - 2AB^*\hat{b}] \quad (89)$$

we have

$$\frac{\partial}{\partial t} \hat{b}^{\dagger n} = i[\dot{\Theta} - m(1+2|A|^2)]n\hat{b}^{\dagger n} - 2imB^*A \left[\frac{n(n-1)}{2} \hat{b}^{\dagger n-2} + \hat{b}^{\dagger n-1} \hat{b} \right] \quad (90)$$

One finally obtains:

$$\begin{aligned} \frac{\partial}{\partial t} |n\rangle_b &= \{ in[\dot{\Theta} - m(2|A|^2 + 1)] + {}_b\langle 0 | \frac{\partial}{\partial t} | 0 \rangle_b \} |n\rangle_b \\ &\quad - imB^*A\sqrt{n(n-1)} |n-2\rangle_b - imBA^*\sqrt{(n+1)(n+2)} |n+2\rangle_b \end{aligned} \quad (91)$$

and

$$\frac{\partial}{\partial t} |\alpha\rangle_b = \left\{ -imBA^*\hat{b}^{\dagger 2} + \alpha \left[i\dot{\Theta} - im(1+2|A|^2) \right] \hat{b}^\dagger + \left[{}_b\langle 0 | \frac{\partial}{\partial t} | 0 \rangle_b - imAB^*\alpha^2 \right] \right\} |\alpha\rangle_b. \quad (92)$$

where

$${}_b\langle 0 | \frac{\partial}{\partial t} | 0 \rangle_b = -i\dot{\varphi}_0 - im(|A|^2 + \frac{1}{2}) \quad (93)$$

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 “The work of R. Brout, F. Englert and E. Gunzig in [1], as well as the subsequent self-consistently driven approach of R. Brout, F. Englert and P. Spindel (*Phys. Rev. Lett.* **43** (1979) 417) did not give full justice to the infra-red sector of the fluctuations of the scalar field. Among other things, rescaling all the fields by the cosmological scale factor leads to surface terms which probably play a very important rôle in the infra-red and hence their neglect results in a distortion of the true physics. The analysis of the fluctuations of the homogeneous mode in the present paper is a case in point. The physics is far richer than envisioned in our original work. “

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